Effective Moduli of Composite Materials in Dynamic Problems

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This article discusses the magnitude of the errors and their influencing parameters in the use of the effective modulus theory to solve dynamic problems of composite materials, based on the development of rigorous solutions of elasticity to the natural axial shear vibration of a laminated composite beam. Exact formulas for the natural frequency and for the vibrating mode in both the fixed and the free boundary cases are worked out. Numerical values of natural frequencies obtained from our solutions are compared to those provided by the effective modulus theory. From the comparison, a number of conclusions of practical significance have been drawn. One of them states that the dynamic effective moduli strongly depend on both the frequency and vibrating mode and therefore can be approximated to simple constants only crudely.

Nomenclature

A = amplitude parameter
G = shear modulus of an isotropic material
G_{xz} = shear modulus of an orthotropic material in the xz plane
G_{yz} = shear modulus of an orthotropic material in the yz plane
S = ply cross section
a = ply thickness
b = beam and ply width
w = space-dependent portion of the harmonic displacement
τ_{xz} = shearing stress in the xz plane
τ_{yz} = shearing stress in the yz plane
Ω = circular frequency
ρ = ply density
∇^2 = the Laplacian

Superscripts
m = non-negative integer

Subscripts
e = effective quantity
N = ply number

Introduction

COMPOSITE materials are usually made up of a large number of constituents with different geometries and properties. As a result, the complex heterogeneity and nonuniformity render the stress analysis of composite materials difficult to deal with, even for static cases.[1] The situation deteriorates further when the dynamic effect of loading is considered. The additional complexity of mass heterogeneity and the presence of a myriad of various waves makes this problem an almost unsurmounted one to be solved exactly. Consequently, in solving the dynamic problem of composite materials, one must usually resort to the effective modulus theory, which reduces the real heterogeneous material to a hypothetic homogeneous one and has been widely used in the stress analysis of composite materials under static loading. Yet it is well known that the effective modulus theory is unable to provide an exact representation of the response of a composite medium to dynamic loading.[2] Therefore, it is of practical interest to study the magnitude of the errors and the influencing parameters that are the consequence of applying the effective modulus theory to dynamic problems of composite materials.

In a preliminary attempt at such an investigation, the most straightforward procedure seems to be to compare some exact solutions of elasticity to dynamic problems of composite materials with the corresponding approximate solutions obtained from the effective modulus theory, and then try to deduce significant conclusions from the comparison. Though there are a number of available solutions to the dynamic problem of composite materials, most of them deal with unbounded or bounded composite media idealized as homogeneous orthotropic bodies.[3-5] There still seem to be few rigorous elasticity solutions to real composite media of finite dimensions under dynamic loading. Hence, in this article we first investigate the natural axial shear vibration of a laminated composite beam and work out the elasticity solution to the problem, then compare the solution to that produced by the effective modulus method. From the comparison, a number of general conclusions on the validity of the effective modulus theory are drawn. Among them it is found that the dynamic effective moduli vary with both the frequency and the vibrating mode and cannot be regarded as simple constants. This deficiency in the effective modulus theory can be used to explain why the theory cannot provide precise solutions to the dynamic problem of composite materials.

Fixed Boundary—Problem and Solution

In this section, we study the natural axial shear vibration of a layered composite beam of a rectangular cross section. The beam is composed of plies of arbitrary orthotropic materials of arbitrary thickness, with one of their material principal directions parallel to the beam axis orientation. On its lateral surface the beam is supported so that the axial displacement along the cross section boundary vanishes.

Using τ_{xz}, τ_{yz}, and w to denote the space-dependent portions of the harmonic shearing stresses and displacement in the problem, respectively, the stress/displacement relations and the equation of motion for the ith ply can be expressed in the following form:

$$\tau_{xzi} = G_{xzi} \frac{\partial w_i}{\partial x} \tag{1a}$$

$$\tau_{yzi} = G_{yzi} \frac{\partial w_i}{\partial y} \tag{1b}$$

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$$G_{xzi}\frac{\partial^2 w_i}{\partial x^2}+G_{yzi}\frac{\partial^2 w_i}{\partial y^2}+\rho_i\Omega^2 w_i=0 \tag{2}$$

For the particular case of an isotropic ply $G_{xzi}=G_{yzi}=G_i$, the equation of motion can be written in the simplified expression[6]

$$\nabla^2 w_i+\frac{\rho_i}{G_i}\Omega^2 w_i=0 \tag{3}$$

For the sake of simplicity, henceforth we will develop our formulation and solution to the problem for a composite beam consisting of isotropic plies. However, as will be shown in the following subsection, the formulation and solution thus developed can also be applied without any essential alterations to composite beams made up of orthotropic plies.

Double-Layer Configuration

In this subsection we solve the problem of the particular case of double-layer configuration (see Fig. 1). The governing equation and the boundary and interface conditions dominating this case can be expressed as follows:

$$\nabla^2 w_i+\frac{\rho_i}{G_i}\Omega^2 w_i=0 \qquad \text{in } S_i,\ i=1,2 \tag{4}$$

On the boundary

$$w_1=w_2=0 \tag{5}$$

On the interface of S_1 and S_2

$$w_1=w_2 \tag{6a}$$

$$G_1=\frac{\partial w_1}{\partial x}=G_2\frac{\partial w_2}{\partial x} \tag{6b}$$

By direct verification we immediately see that the following formulas

$$w_1=A_1^m\sin\frac{m\pi y}{b}\sin\frac{m_1\pi(x+a_1)}{a_1} \tag{7}$$

$$w_2=A_2^m\sin\frac{m\pi y}{b}\sin\frac{m_2\pi(x-a_2)}{a_2} \tag{8}$$

satisfy Eqs. (4) and (5) provided we take $m=1,2,3,\ldots$ and

$$\frac{\rho_i}{G_i}\Omega^2=\left(\frac{m\pi}{b}\right)^2+\left(\frac{m_i\pi}{a_i}\right)^2, \qquad i=1,2 \tag{9}$$

To satisfy the interface condition we substitute Eqs. (7) and (8) into Eq. (6) to obtain

$$A_1^m\sin m_1\pi+A_2^m\sin m_2\pi=0 \tag{10}$$

$$m_1a_2G_1A_1^m\cos m_1\pi-m_2a_1G_2A_2^m\cos m_2\pi=0 \tag{11}$$

Utilizing Eqs. (9) and (10), Eq. (11) can be rewritten in the following form suitable for frequency evaluation:

$$\begin{aligned}&G_1\sqrt{(\rho_1/G_1)\Omega^2-(m\pi/b)^2}\tan[a_2\sqrt{(\rho_2/G_2)\Omega^2-(m\pi/b)^2}]\\&+G_2\sqrt{(\rho_2/G_2)\Omega^2-(m\pi/b)^2}\\&\times\tan[a_1\sqrt{(\rho_1/G_1)\Omega^2-(m\pi/b)^2}]=0\end{aligned} \tag{12}$$

Using Eqs. (9–11) or (12), all unknowns in the problem can be determined except the amplitude parameter A_1^m or A_2^m which can be of arbitrary nonzero value.

Note that for small frequencies the quantities under the square root in Eq. (12) may take negative values. In such cases it is preferable to rewrite Eq. (12) using complex function relations to avoid complex variable computations. For instance, for $(\rho_2/G_2)\Omega^2>(m\pi/b)^2>(\rho_1/G_1)\Omega^2$ Eq. (12) can be rewritten as

$$\begin{aligned}&G_1\sqrt{(m\pi/b)^2-(\rho_1/G_1)\Omega^2}\tan[a_2\sqrt{(\rho_2/G_2)\Omega^2-(m\pi/b)^2}]\\&+G_2\sqrt{(\rho_2/G_2)\Omega^2-(m\pi/b)^2}\\&\times\tanh[a_1\sqrt{(m\pi/b)^2-(\rho_1/G_1)\Omega^2}]=0\end{aligned} \tag{13}$$

In case the composite beam is laminated by orthotropic plies, the appropriate governing equation, boundary, and interface conditions can be put into the following form:

$$\frac{\partial^2 w_i}{\partial \bar{x}^2}+\frac{\partial^2 w_i}{\partial y^2}+\frac{\rho_i}{G_{xzi}}\Omega^2 w_i=0 \qquad \text{in } S_i,\ i=1,2 \tag{14}$$

On the boundary

$$w_1=w_2=0 \tag{15}$$

On the interface

$$w_1=w_2 \tag{16a}$$

$$\bar{G}_{xz1}\frac{\partial w_1}{\partial \bar{x}}=\bar{G}_{xz2}\frac{\partial w_2}{\partial \bar{x}} \tag{16b}$$

where

$$\bar{x}=\sqrt{G_{yz1}/G_{xz1}}\,x \quad \text{for} \quad -a_1\le x<0$$

$$\bar{x}=\sqrt{G_{yz2}/G_{xz2}}\,x \quad \text{for} \quad 0\le x\le a_2$$

$$\bar{G}_{xzi}=\sqrt{G_{xzi}/G_{yzi}}\,G_{xzi}, \qquad i=1,2$$

Equations (14–16) are identical to Eqs. (4–6) for isotropic plies in mathematical form. We then see that in the formulation and solution to our problem an orthotropic ply can be treated just as an isotropic one, only its thickness a_i should be replaced by $\sqrt{(G_{yzi})/(G_{xzi})}a_i$ and its shear modulus G_{xzi} by $\sqrt{(G_{xzi})/(G_{yzi})}G_{xzi}$. The preceding analysis then justifies our preference for isotropic plies in this investigation.

Multilayer Configuration

In this situation (see Fig. 2) the governing equation and the boundary and interface conditions are expressed by the following equations:

$$\nabla^2 w_i+\frac{\rho_i}{G_i}\Omega^2 w_i=0 \qquad \text{in } S_i,\ i=1,2,\ldots,N \tag{17}$$

On the boundary

$$w_1=w_2=\cdots=w_N=0 \tag{18}$$

On the $i-1$, i interface

$$w_{i-1}=w_i \tag{19a}$$

$$G_{i-1}\frac{\partial w_{i-1}}{\partial x}=G_i\frac{\partial w_i}{\partial x}, \qquad i=2,3,\ldots,N-1 \tag{19b}$$

We propose the solutions of w_i in the problem such that

$$w_1=A_1^m\sin\frac{m\pi y}{b}\sin\frac{m_1\pi(x+a_1)}{a_1} \tag{20}$$

$$w_2 = A^m_{2(1)} \sin \frac{m\pi y}{b} \sin \frac{m_2\pi(x-a_2)}{a_2} + A^m_{2(2)} \sin \frac{m\pi y}{b} \sin \frac{m_2 \pi x}{a_2} \tag{21}$$

$$w_i = A^m_{i(1)} \sin \frac{m\pi y}{b} \sin \frac{m_i\pi[x-(a_2+a_3+\ldots+a_i)]}{a_i} + A^m_{i(2)} \sin \frac{m\pi y}{b} \sin \frac{m_i\pi[x-(a_2+a_3+\ldots+a_{i-1})]}{a_i},$$

$$i=3,4,\ldots,N-1 \tag{22}$$

$$w_N = A^m_N \sin \frac{m\pi y}{b} \sin \frac{m_N\pi[x-(a_2+a_3+\ldots+a_N)]}{a_N} \tag{23}$$

Following a procedure similar to that described in the preceding subsection, it can be seen that the above displacement functions satisfy all the aforementioned requirements, provided we assume $m=1,2,3,\ldots$, determine m_i using the following formula

$$\frac{\rho_i}{G_i}\Omega^2 = \left(\frac{m\pi}{b}\right)^2 + \left(\frac{m_i\pi}{a_i}\right)^2, \qquad i=1,2,\ldots,N \tag{24}$$

and find $A^m_1, A^m_{2(1)}, A^m_{2(2)}, \ldots, A^m_N$ through the following system of equations:

$$A^m_1 \sin m_1\pi = -A^m_{2(1)} \sin m_2\pi \tag{25}$$

$$G_1 A^m_1 \frac{m_1\pi}{a_1} \cos m_1\pi = G_2\left(A^m_{2(1)} \frac{m_2\pi}{a_2} \cos m_2\pi + A^m_{2(2)} \frac{m_2\pi}{a_2}\right) \tag{26}$$

$$A^m_{i(2)} \sin m_i\pi = -A^m_{i+1(1)} \sin m_{i+1}\pi, \qquad i=2,3,\ldots,N-2 \tag{27}$$

$$G_i\left(A^m_{i(1)} \frac{m_i\pi}{a_i} + A^m_{i(2)} \frac{m_i\pi}{a_i} \cos m_i\pi\right) = G_{i+1}\left(A^m_{i+1(1)} \frac{m_{i+1}\pi}{a_{i+1}} \cos m_{i+1}\pi + A^m_{i+1(2)} \frac{m_{i+1}\pi}{a_{i+1}}\right),$$

$$i=2,3,\ldots,N-2 \tag{28}$$

$$A^m_{N-1(2)} \sin m_{N-1}\pi = -A^m_N \sin m_N\pi \tag{29}$$

$$G_{N-1}\left(A^m_{N-1(1)} \frac{m_{N-1}\pi}{a_{N-1}} + A^m_{N-1(2)} \frac{m_{N-1}\pi}{a_{N-1}} \cos m_{N-1}\pi\right) = G_N\left(A^m_N \frac{m_N\pi}{a_N} \cos m_N\pi\right) \tag{30}$$

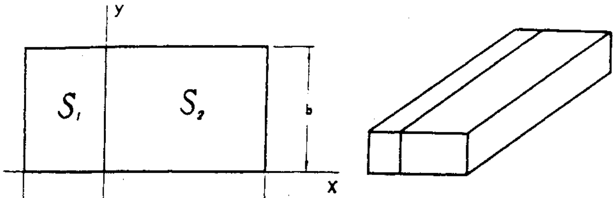

Fig. 1 Double-layer configuration.

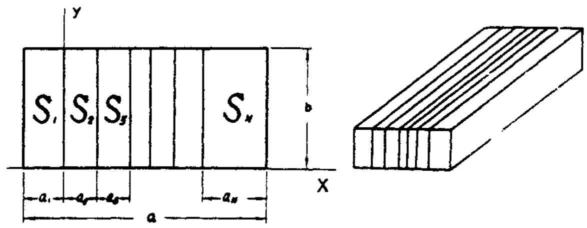

Fig. 2 Multilayer configuration.

Table 1 Numerical values of natural frequency parameter $(\rho_0\Omega^2)/(G_2\pi^2)$ for $G_1/G_2=2$

			$m=1$			
	n	1	2	3	4	5
	2	0.703	1.676	3.540	5.738	8.990
	4	0.708	1.801	3.462	5.595	8.923
	6	0.708	1.696	3.619	5.940	8.943
	8	0.708	1.703	3.313	6.162	9.174
N	10	0.708	1.705	3.457	5.540	9.431
	12	0.708	1.706	3.358	5.628	8.364
	14	0.708	1.706	3.364	5.659	8.538
	16	0.708	1.707	3.367	5.674	8.601
	18	0.708	1.707	3.369	5.683	8.633
	20	0.708	1.707	3.370	5.689	8.652
Ωe[a]		0.667	1.667	3.333		
Er, %[b]		5.8	2.3	1.1		
Ge/G_2[c]		1.416	1.366	1.348		
			$m=2$			
	n	1	2	3	4	5
	2	1.618	2.841	4.563	6.869	9.990
	4	1.803	2.814	4.527	6.706	10.000
	6	1.822	2.833	4.655	6.970	13.229
	8	1.828	2.834	4.480	7.206	10.177
N	10	1.830	2.834	4.491	6.731	10.479
	12	1.831	2.834	4.495	6.785	9.576
	14	1.832	2.834	4.497	6.804	9.707
	16	1.832	2.834	4.498	6.813	9.754
	18	1.833	2.834	4.498	6.819	9.778
	20	1.833	2.834	4.499	6.822	9.792
Ωe[a]		1.667	2.667	4.333		
Er, %[b]		9.1	5.9	3.7		
Ge/G_2[c]		1.466	1.417	1.385		
			$m=3$			
	n	1	2	3	4	5
	2	2.982	4.729	6.379	8.746	11.736
	4	3.503	4.403	6.377	8.621	11.805
	6	3.630	4.652	6.331	8.743	11.791
	8	3.666	4.685	6.374	8.917	11.888
N	10	3.682	4.695	6.375	8.681	11.206
	12	3.691	4.700	6.375	8.696	11.569
	14	3.696	4.702	6.375	8.701	11.642
	16	3.699	4.704	6.375	8.703	11.667
	18	3.701	4.705	6.375	8.703	11.679
	20	3.702	4.706	6.375	8.703	11.687
Ωe[a]		3.333	4.333	6.000	8.333	
Er, %[b]		10.0	7.9	5.9	4.3	
Ge/G_2[c]		1.481	1.448	1.417	1.393	

[a]Natural frequency parameter computed from the static effective modulus. [b]The magnitude of error computed from the formula Er = (stationary value of frequency − Ωe)/stationary value of frequency. [c]Dynamic effective modulus computed from the stationary value of frequency.
Note: the above three items are listed only for those cases where the stationary value of frequency has or nearly has been reached on or before $N=20$.

In total, there are $3N-1$ unknowns [Ω; $m_1,m_2,\ldots,m_N$; A_1^m, $A_{2(1)}^m$, $A_{2(2)}^m,\ldots,A_N$] involved in the problem; among them, $3N-2$ are independent ones. (One of the amplitude parameters, A_1^m, for example, can be of arbitrary nonzero value.) The independent unknowns can be determined from the $3N-2$ equations, already enumerated. Hence, our solution to the problem is well developed and the reduced natural frequency equation can now be derived. By way of illustration, for the case of $N=3$ the reduced natural frequency equation reads

$$\frac{\sin m_1\pi}{\sin m_2\pi}=\left\{\left[\frac{G_1m_1a_2}{G_2m_2a_1}\cos m_1\pi\left(\cos m_2\pi+\frac{G_3m_3a_2}{G_2m_2a_3}\frac{\sin m_2\pi}{\sin m_3\pi}\cos m_3\pi\right)\right]\Big/\left[\sin^2 m_2\pi-\frac{G_3m_3a_2}{G_2m_2a_3}\frac{\sin m_2\pi}{\sin m_3\pi}\cos m_2\pi\cos m_3\pi\right]\right\} \tag{31}$$

When N becomes larger, the reduced natural frequency equation takes a very complicated form. In this situation it is suitable to keep the natural frequency equations in their original form expressed by Eqs. (24–30).

Free Boundary—Problem and Solution

In this section, we discuss the natural axial shear vibration of the laminated composite beam with free lateral surface such that the shearing stresses along the cross section boundary vanish. The corresponding solution can be sought in a similar way as described in the preceding section for beams with fixed surfaces.

Double-Layer Configuration

In this case the governing equation and the interface conditions, [Eqs. (4) and (6)], remain unchanged, but the boundary conditions should be replaced by

$$\frac{\partial w_1}{\partial y}=\frac{\partial w_2}{\partial y}=0,\qquad y=0,b,\quad -a_1\le x\le a_2 \tag{32}$$

$$\frac{\partial w_1}{\partial x}=\frac{\partial w_2}{\partial x}=0,\qquad x=-a_1,a_2,\quad 0\le y\le b \tag{33}$$

To satisfy these requirements we propose the displacement functions in the following form:

$$w_1=A_1^m\cos\frac{m\pi y}{b}\cos\frac{m_1\pi(x+a_1)}{a_1} \tag{34}$$

$$w_2=A_2^m\cos\frac{m\pi y}{b}\cos\frac{m_2\pi(x-a_2)}{a_2} \tag{35}$$

It can be easily verified that the above equations represent the exact solution to our problem, provided we take $m=0,1,2,\ldots$ and determine the remaining unknowns in Eqs. (34) and (35) from the following equations:

$$\frac{\rho_i}{G_i}\Omega^2=\left(\frac{m\pi}{b}\right)^2+\left(\frac{m_i\pi}{a_i}\right)^2,\qquad i=1,2 \tag{36}$$

$$A_1^m\cos m_1\pi-A_2^m\cos m_2\pi=0 \tag{37}$$

$$m_1a_2G_1A_1^m\sin m_1\pi+m_2a_1G_2A_2^m\sin m_2\pi=0 \tag{38}$$

The reduced natural frequency equation can be deduced and expressed by

$$G_1\sqrt{(\rho_1/G_1)\Omega^2-(m\pi/b)^2}\tan[a_1\sqrt{(\rho_1/G_1)\Omega^2-(m\pi/b)^2}]+G_2\sqrt{(\rho_2/G_2)\Omega^2-(m\pi/b)^2}\times\tan[a_2\sqrt{(\rho_2/G_2)\Omega^2-(m\pi/b)^2}]=0 \tag{39}$$

Multilayer Configuration

As in the case of double-layer configurations, the governing equation and the interface conditions [Eqs. (17) and (19)] remain unchanged, while the boundary condition should be replaced by

$$\frac{\partial w_i}{\partial y}=0,\qquad y=0,b\quad -a_1\le x\le a_N;\quad i=1,2,\ldots,N \tag{40}$$

$$\frac{\partial w_i}{\partial x}=0,\qquad x=-a_1,a_N,\quad 0\le y\le b;\quad i=1,N \tag{41}$$

We now propose the solution of w_i such that

$$w_1=A_1^m\cos\frac{m\pi y}{b}\cos\frac{m_1\pi(x+a_1)}{a_1} \tag{42}$$

$$w_2=A_{2(1)}^m\cos\frac{m\pi y}{b}\cos\frac{m_2\pi(x-a_2)}{a_2}+A_{2(2)}^m\cos\frac{m\pi y}{b}\cos\frac{m_2\pi x}{a_2} \tag{43}$$

$$w_i=A_{i(1)}^m\cos\frac{m\pi y}{b}\cos\frac{m_i\pi[x-(a_2+a_3+\ldots+a_i)]}{a_i}+A_{i(2)}^m\cos\frac{m\pi y}{b}\cos\frac{m_i\pi[x-(a_2+a_3+\ldots+a_{i-1})]}{a_i},\qquad i=3,4,\ldots,N-1 \tag{44}$$

$$w_N=A_N^m\cos\frac{m\pi y}{b}\cos\frac{m_N\pi[x-(a_2+a_3+\ldots+a_N)]}{a_N} \tag{45}$$

Following a procedure similar to that described in the foregoing section, we can assure that the preceding displacement functions satisfy all the requirements mentioned provided we take $m=0,1,2,\ldots$, determine m_i from the following formula:

$$\frac{\rho_i}{G_i}\Omega^2=\left(\frac{m\pi}{b}\right)^2+\left(\frac{m_i\pi}{a_i}\right),\qquad i=1,2,\ldots,N \tag{46}$$

and find $A_1^m,A_{2(1)}^m,A_{2(2)}^m,\ldots,A_N^m$ through the following system of equations:

$$A_1^m\cos m_1\pi=A_{2(1)}^m\cos m_2\pi+A_{2(2)}^m \tag{47}$$

$$A_1^m\frac{m_1}{a_1}\sin m_1\pi=-A_{2(1)}^m\frac{m_2}{a_2}\sin m_2\pi \tag{48}$$

$$A_{i(1)}^m+A_{i(2)}^m\cos m_i\pi=A_{i+1(1)}^m\cos m_i\pi+A_{i+1(2)}^m,\qquad i=2,3,\ldots,N-2 \tag{49}$$

Table 2 Numerical values of natural frequency parameter $(\rho_0\Omega^2)/(G_2\pi^2)$ for $G_1/G_2=3$

	$m=1$				
n	1	2	3	4	5
N 2	0.840	1.970	4.108	6.986	10.178
4	0.875	2.238	4.068	6.370	10.583
6	0.875	1.975	4.533	7.053	10.372
8	0.875	1.989	3.734	7.739	11.008
10	0.875	1.994	3.809	6.123	11.860
12	0.875	1.996	3.835	6.311	9.119
14	0.875	1.997	3.848	6.381	9.475
16	0.875	1.998	3.855	6.417	9.618
18	0.875	1.999	3.860	6.438	9.692
20	0.875	1.999	3.863	6.452	9.737
Ωe[a]	0.750	1.875	3.750		
Er, %[b]	14.3	6.6	3.0		
Ge/G_2[c]	1.750	1.599	1.547		

	$m=2$				
n	1	2	3	4	5
N 2	1.757	3.484	5.365	8.385	11.530
4	2.256	3.359	5.412	7.879	11.902
6	2.330	3.494	5.720	8.283	11.685
8	2.351	3.498	5.342	8.953	12.158
10	2.360	3.499	5.360	7.793	13.086
12	2.365	3.499	5.366	7.899	10.835
14	2.368	3.500	5.369	7.937	11.096
16	2.370	3.500	5.371	7.956	11.197
18	2.371	3.500	5.372	7.968	11.248
20	2.372	3.500	5.373	7.975	11.280
Ωe[a]	1.875	3.000	4.875		
Er, %[b]	26.5	14.3	9.3		
Ge/G_2[c]	1.898	1.750	1.653		

	$m=3$				
n	1	2	3	4	5
N 2	3.086	5.399	7.961	10.585	13.995
4	4.159	4.991	7.889	10.474	14.129
6	4.584	5.758	7.557	10.535	13.981
8	4.719	5.899	7.852	10.888	14.207
10	4.778	5.942	7.865	10.470	15.072
12	4.809	5.962	7.869	10.486	13.626
14	4.827	5.973	7.871	10.491	13.751
16	4.838	5.980	7.872	10.494	13.797
18	4.846	5.984	7.873	10.496	13.820
20	4.852	5.988	7.874	10.497	13.833
Ωe[a]		4.875	6.750	9.375	
Er, %[b]		18.7	13.9	10.7	
Ge/G_2[c]		1.844	1.750	1.680	

[a]Natural frequency parameter computed from the static effective modulus. [b]The magnitude of error computed from the formula Er = (stationary value of frequency $-\Omega e$)/stationary value of frequency. [c]Dynamic effective modulus computed from the stationary value of frequency.
Note: the above three items are listed only for those cases where the stationary value of frequency has or nearly has been reached on or before $N=20$.

Table 3 Numerical values of natural frequency parameter $(\rho_0\Omega^2)/(G_2\pi^2)$ for $G_1/G_2=5$

	$m=1$				
n	1	2	3	4	5
N 2	0.992	2.500	4.646	8.582	12.500
4	1.159	2.846	4.785	7.703	12.908
6	1.164	2.384	5.861	8.381	11.872
8	1.166	2.403	4.273	10.069	13.230
10	1.166	2.409	4.389	6.790	15.476
12	1.166	2.412	4.432	7.084	9.915
14	1.167	2.413	4.453	7.204	10.460
16	1.167	2.414	4.466	7.267	10.700
18	1.167	2.415	4.474	7.304	10.830
20	1.167	2.415	4.479	7.329	10.910
Ωe[a]	0.833	2.083	4.167		
Er, %[b]	28.6	13.7	7.2		
Ge/G_2[c]	2.334	1.932	1.796		

	$m=2$				
n	1	2	3	4	5
N 2	1.861	4.212	6.728	10.000	14.633
4	2.945	3.969	6.760	10.000	14.501
6	3.234	4.592	7.115	13.712	18.678
8	3.319	4.636	6.735	11.382	14.617
10	3.356	4.649	6.743	9.408	16.819
12	3.375	4.655	6.746	9.536	12.632
14	3.387	4.659	6.748	9.584	12.991
16	3.394	4.661	6.748	9.609	12.139
18	3.399	4.662	6.749	9.624	13.217
20	3.402	4.663	6.749	9.633	13.265
Ωe[a]	2.083	3.333	5.417		
Er, %[b]	38.8	28.5	19.7		
Ge/G_2[c]	2.725	2.332	2.077		

	$m=3$				
n	1	2	3	4	5
N 2	3.158	5.819	9.842	13.423	17.555
4	4.893	5.512	10.473	13.400	17.424
6	6.088	7.345	8.930	13.417	17.184
8	6.576	7.950	10.231	13.376	17.427
10	6.797	8.149	10.382	13.417	18.920
12	6.914	8.241	10.429	13.417	16.954
14	6.983	8.292	10.452	13.417	17.059
16	7.027	8.323	10.465	13.417	17.098
18	7.057	8.344	10.473	13.417	17.118
20	7.078	8.358	10.479	13.417	17.130
Ωe[a]			7.500	10.417	
Er, %[b]			28.5	22.4	
Ge/G_2[c]			2.332	2.147	

[a]Natural frequency parameter computed from the static effective modulus. [b]The magnitude of error computed from the formula Er = (stationary value of frequency $-\Omega e$)/stationary value of frequency. [c]Dynamic effective modulus computed from the stationary value of frequency.
Note: the above three items are listed only for those cases where the stationary value of frequency has or nearly has been reached on or before $N=20$.

$$A_{i(2)}^{m}\frac{m_i}{a_i}\sin m_i\pi=A_{i+1(1)}^{m}\frac{m_{i+1}}{a_{i+1}}\sin m_{i+1}\pi,\qquad i=2,3,\ldots,N-2 \tag{50}$$

$$A_{N-1(1)}^{m}+A_{N-1(2)}^{m}\cos m_{N-1}\pi=A_N^m\cos m_N\pi \tag{51}$$

$$A_{N-1(2)}^{m}\frac{m_{N-1}}{a_{N-1}}\sin m_{N-1}\pi=A_N^m\frac{m_N}{a_N}\sin m_N\pi \tag{52}$$

These formulas can be used to determine the frequency and the vibrating mode of the laminated beam with ply number $N>2$.

Results and Conclusions

We have obtained the elasticity solution to the problem of natural axial shear vibration of composite beams. Accurate values of natural frequencies and closed forms of vibrating modes can be gained from the solution. In Tables 1-3 we presented numerical values of natural frequencies for a fixed composite beam of periodically alternative plies of uniform thickness (i.e., $G_1=G_3=\ldots=G_{N-1}$, $G_2=G_4=\ldots=G_N$, $a_1=a_2=\ldots=a_N$) in the particular case of $b=2$, $a_i=2/N$, $G_1=2G_2$, $3G_2$, $5G_2$, and $\rho_i=\rho_0=$ const. We used the last identity to exclude the mass nonuniformity to be coupled with the modulus heterogeneity, which is the main concern of this paper. From the numerical data presented in the tables, the following trends can be observed and the following general conclusions can be drawn.

As the ply number of the composite beam increases in accordance with decreasing ply thickness, the natural frequency changes its value rather irregularly for small N; on the other hand, a stationary value of frequency is definitely reached when N becomes sufficiently large. From this phenomenon we see immediately that the effective modulus theory is unfeasible and senseless for composite structural components composed of only a small number of constituents. Hence, later on we will concentrate our discussion on composed beams having a sufficiently large number of plies and on stationary values of frequencies.

Next we see if the effective modulus theory can provide good approximate results to our problem. The static effective modulus in this investigation is expressed by $2G_1G_2/(G_1+G_2)$,[7] and the natural frequency formula for a homogeneous, isotropic, rectangular beam is available and can be written as[8]

$$\frac{\rho}{G\pi^2}\,\Omega^2=\left(\frac{m^2}{b^2}+\frac{n^2}{a^2}\right),\qquad m,n=1,2,3,\ldots \tag{53}$$

where a and b are the cross-sectional dimensions of the homogeneous beam and ρ, G, and Ω represent its density, shear modulus, and circular frequency, respectively. Using the static effective modulus and Eq. (53), the values of the natural frequencies computed from the effective modulus theory are obtained and listed in Tables 1-3. A comparison of frequency values obtained from the elasticity solution with those obtained from the static effective modulus method is then made, and the result shows that the magnitude of errors incurred by the static effective modulus enlarges as the ratio G_1/G_2 increases. This implies that the effective modulus theory cannot work accurately when the two constitutents of the composite beam have dramatically different material properties. It can also be seen that for a definite value of G_1/G_2, the magnitude of errors enlarges with increasing m under a fixed value of n but decreases with increasing n under a fixed value of m. This phenomenon shows that the effective moduli of composite materials in dynamic problems strongly depend on both the frequency and mode shape in a very complicated manner. Presenting it as a single constant or even a simple function (e.g., a function whose value monotonously increases with the frequency) is not sufficient for the effective modulus theory to be a reasonably precise one in solving dynamic problems of composite materials.

We can see from the tables that the effective modulus theory based on the static effective modulus can provide adequate accurate results only in the limited scope in which G_1/G_2, m, and n are all small numbers. In another way of applying the effective modulus theory, we can find the exact value of the dynamic effective modulus in the computation of a particular natural frequency, via substituting the stationary value of the frequency into Eq. (53). It is shown in Tables 1-3 that the dynamic effective modulus thus obtained does not stay constant, but varies accordingly with the magnitude of the errors incurred by the effective modulus theory. This fact again demonstrates that the effective moduli can only be approximately represented by simple constants.

We have analyzed the magnitude of the errors and their influencing parameters in the use of effective modulus theory to solve dynamic problems of composite materials; we also have found that the stationary values of natural frequencies are important quantities for characterizing the natural vibration of composite materials. Therefore, a more adequate form of the dynamic effective modulus theory should be developed in association with the stationary value of natural frequency.

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